

ON EQUIFOCAL FINSLER SUBMANIFOLDS AND ANALYTIC MAPS*

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ABSTRACT

A relevant property of equifocal submanifolds is that their parallel sets are still immersed submanifolds, which makes them a natural generalization of the so-called isoparametric submanifolds. In this paper, we prove that the regular fibers of an analytic map $\pi : M^{m+k} \rightarrow B^k$ are equifocal whenever M^{m+k} is endowed with a complete Finsler metric and there is a restriction of π which is a Finsler submersion for a certain Finsler metric on the image. In addition, we prove that when the fibers provide a singular foliation on M^{m+k} , then this foliation is Finsler.

1. Introduction

Roughly speaking, as the name might suggest, an equifocal submanifold L in a Riemannian manifold M is one whose “parallel sets” are always (immersed) submanifolds, even when they contain focal points of L (see the formal definition below).

From Somigliana’s article on Geometric Optics in 1919, and Segre and Cartan’s work in the 1930s to the present day, this natural concept received different names and was treated in different ways, but it has been playing an important role in the theory of submanifolds and problems with symmetries; for surveys on the relation between the theory of isoparametric submanifolds and isometric actions see Thorbergsson’s surveys [41, 42, 43] and for books on these topics see, e.g., [4, 11, 35].

Isoparametric submanifolds and regular orbits of isometric actions are examples of equifocal submanifolds. Both kinds of submanifolds are leaves of the so-called singular Riemannian foliations (SRF for short), i.e., singular foliations whose leaves are locally equidistant; see e.g., [5]. On the other hand, the regular leaves of an SRF are equifocal (see [6]). Therefore, so far, equifocality and SRF’s are closely related.

Also over the past decades, important examples have been presented as a pre-image of regular values of analytic maps; see [15] and [38]. And there are good reasons for that. For example, as proved in [30], the leaves of SRF’s with closed leaves in Euclidean spaces are always pre-images of polynomial maps.

Several problems related to symmetries and wavefronts presented in Riemannian geometry are also natural problems in Finsler geometry. For example, the Finsler distance (a particular case of transnormal functions) has been used to model forest wildfires and Huygens’ principle; see [32] and also [12, 26]. This is

one of the several motivations for a recent development of Finsler's isoparametric theory, and again isoparametric and transnormal analytic functions naturally appear; see [2, 13, 19, 20, 21, 22, 47]. For other applications see [14].

Very recently we have laid in [3] the foundations of the new theory of singular Finsler foliations (SFF), i.e., a singular foliation where the leaves are locally equidistant (but the distance from a plaque or leaf L_a to L_b does not need to be equal to the distance from a plaque or leaf L_b to L_a); see Lemma 2.5. The partition of a Finsler manifold into orbits of a Finsler action is a natural example of SFF. We have also presented non-homogeneous examples using analytic maps [3, Example 2.13]. Among other results, we proved the equifocality of the regular leaves of an SFF on a Randers manifold, where the wind is an infinitesimal homothety. Nevertheless, to prove the equifocality of the regular leaves of an SFF in the general case becomes a challenging problem, presenting new aspects that did not appear in the analogous problem in Riemannian geometry.

For all these reasons, it is natural to ask if we can prove the equifocality of leaves of SFF's described by analytic maps. In this article, we tackle this issue proving it under natural conditions. During our investigation we generalize some Riemannian techniques (like Wilking's distribution) and obtain some results that, as far as we know, do not appear in the literature even in the Riemannian case.

Throughout this article, (M, F) will always be a complete analytical Finsler manifold.

In order to formalize the concept of "parallel sets" and equifocality, let us start by recalling the definition of equifocal hypersurfaces.

Let L be an (oriented) hypersurface with normal unit vector field ξ (in the non-reversible Finsler case, there could be two independent normal unit vector fields to L). We define the **endpoint map** $\eta_\xi : L \rightarrow M$ as

$$\eta_\xi^r(p) = \gamma_{\xi_p}(r),$$

where γ_{ξ_p} is the unique geodesic with $\dot{\gamma}_{\xi_p}(0) = \xi_p$. In this case the parallel sets are

$$L_r = \eta_\xi^r(L).$$

The hypersurface L is said to be **equifocal** if $d\eta_\xi^r$ has constant rank for all r . From the constant rank theorem, it is not difficult to check that if L is compact, then the parallel sets L_r are immersed submanifolds (with possible intersections).

If the codimension of L is greater than 1, we need to clarify which normal unit vector field ξ along L we consider as there are many different choices. We will follow the same approach as in the Riemannian case.

When L is a regular leaf of an SFF denoted by \mathcal{F} (i.e., L has maximal dimension), there exists a neighborhood U (in M) of the point $p \in L$ and a Finsler submersion $\rho : U \rightarrow S$ so that $\mathcal{F}_U = \mathcal{F} \cap U$ are the pre-images of the map ρ . In this case, ξ can be chosen as a projectable normal vector field along $L \cap U$ with respect to ρ , i.e., an **\mathcal{F} -local basic vector field**. Now, we say that L is an **equifocal submanifold** if, for each local \mathcal{F} -basic unit vector field ξ along L , the differential map $d\eta_\xi^r$ has constant rank for all r .

Finally when $\pi : M \rightarrow B$ is an analytic map, c is a regular value and $L = \pi^{-1}(c)$, ξ can be chosen as a basic vector field along L , i.e., **π -basic**. Once we want to consider “parallel sets” of codimension bigger than one, it is natural to expect that at least near L the fibers are equidistant. Therefore, in this case we will demand that there exists a saturated neighborhood \tilde{U} of L where π is regular and the fibers of π restricted to \tilde{U} are the leaves of a regular Finsler foliation. In other words, the restriction $\pi_{\tilde{U}} : \tilde{U} \rightarrow \pi(\tilde{U}) \subset B$ is a Finsler submersion for some Finsler metric on $\pi(\tilde{U})$. With this assumption, we will say that $L = \pi^{-1}(c)$ is an equifocal submanifold if for each π -basic unit vector field ξ along L , the differential map $d\eta_\xi^r$ has constant rank for all r . Observe that in this definition, the partition $\mathcal{F}_\pi = \{\pi^{-1}(c)\}_{c \in B}$ of M by the fibers of π is not required to be a singular foliation.

THEOREM 1.1: *Let (M, F) be an analytic connected complete Finsler manifold and $\pi : M^{m+k} \rightarrow B^k$ an analytic map between analytic manifolds. Assume that the pre-image $\pi^{-1}(d)$ is always a connected set and that there exists an open subset U where π is regular and such that $\mathcal{F}_\pi|_U$ is a Finsler foliation. We have that:*

- (a) *If $q \in \pi^{-1}(c)$ is a regular point (i.e., $d\pi_q$ is surjective), then all points of $\pi^{-1}(c)$ are regular, i.e., c is a regular value, and the restriction of \mathcal{F}_π to the set of regular points of π is a Finsler foliation.*
- (b) *Each regular level set $\pi^{-1}(c)$ is an equifocal submanifold. In addition, the image of the endpoint map $\eta_\xi^r : \pi^{-1}(c) \rightarrow M$ (for r and ξ fixed) is contained in a level set.*
- (c) *If \mathcal{F}_π is a smooth singular foliation, then \mathcal{F}_π is a singular Finsler foliation.*

Remark 1.2: Item (a) shows how the hypothesis influences the singular fibers. In fact, consider $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as $f(x_1, x_2) = x_1 x_2$. Note that $f^{-1}(0)$ is the stratified set $\{x_2 = 0\} \cup \{x_1 = 0\}$. Also note that each point $p \in f^{-1}(0)$ different from $(0, 0)$ is a regular point, i.e.,

$$\nabla f(p) \neq (0, 0).$$

In other words, in this very natural and simple example item (a) is not fulfilled.

Remark 1.3: In the particular case that the partition of the pre-images of π is already a submetry, and M is Riemannian, then the proof of equifocality follows from [33, Section 3.4]. For other important results on Riemannian submetries see [28].

There are a few interesting open questions. The first one is under which hypotheses can one ensure that the singular set is not just a stratification, but a submanifold. This problem was solved in [2], in the case where the map was an analytic function $f : M \rightarrow \mathbb{R}$. The second question is even when the singular sets are submanifolds, and the fibers are equidistant, under which conditions $\mathcal{F}_\pi = \{\pi^{-1}(c)\}_{c \in B}$ will be a smooth singular foliation, i.e., for each $p \in \pi^{-1}(d)$ and $v \in T_p(\pi^{-1}(d))$, there is a smooth vector field X tangent to the fibers with

$$X_p = v.$$

This paper is organized as follows. In §2, we briefly review a few facts on Finsler geometry and Finsler submersions. In §3, we review the construction of Wilkings' distribution in the Finsler case. In §4, we present the proof of Theorem 1.1. In §5, we discuss the definition of the curvature tensor (related to the definition of Jacobi tensor presented in §2) and some proofs of results (presented in §3) on Jacobi fields. In this way, we hope to make the paper accessible also to readers without previous training in Finsler geometry, or who are more interested in understanding the Riemannian case of the theorem.

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2. Preliminaries

In this section we fix some notations and briefly review a few facts about Finsler Geometry that will be used in this paper. For more details see [9, 40, 39].

2.1. **FINSLER METRICS.** Let M be a manifold. We say that a continuous function $F : TM \rightarrow [0, +\infty)$ is a **Finsler metric** if

- (1) F is smooth on $TM \setminus \mathbf{0}$,
- (2) F is positive homogeneous of degree 1, that is, $F(\lambda v) = \lambda F(v)$ for every $v \in TM$ and $\lambda > 0$,
- (3) for every $p \in M$ and $v \in T_p M \setminus \{0\}$, the **fundamental tensor** of F defined as

$$g_v(u, w) = \frac{1}{2} \frac{\partial^2}{\partial t \partial s} F^2(v + tu + sw)|_{t=s=0}$$

for any $u, w \in T_p M$ is a nondegenerate positive-definite bilinear symmetric form.

In particular, if M is a vector space V , and $F : V \rightarrow \mathbb{R}$ is a function smooth on $V \setminus \{\mathbf{0}\}$ and satisfying the properties (2) and (3) above, then (V, F) is called a **Minkowski space**.

Recall that the **Cartan tensor** associated with the Finsler metric F is defined as

$$C_v(w_1, w_2, w_3) := \frac{1}{4} \frac{\partial^3}{\partial s_3 \partial s_2 \partial s_1} F^2 \left(v + \sum_{i=1}^3 s_i w_i \right) \Big|_{s_1=s_2=s_3=0}$$

for every $p \in M$, $v \in T_p M \setminus \mathbf{0}$, and $w_1, w_2, w_3 \in T_p M$. By the homogeneity of F , it holds that

$$C_v(v, w_1, w_2) = C_v(w_1, v, w_2) = C_v(w_1, w_2, v) = 0.$$

2.2. THE CHERN CONNECTION AND THE INDUCED COVARIANT DERIVATIVE.

In §5.1, we will discuss the concept of anisotropic connection, which means that the Christoffel symbols are functions on the tangent bundle. Here we just briefly review the concept of Chern connection associated with a Finsler metric as a family of affine connections ∇^V , with V a vector field without singularities on an open subset Ω (see [40, Eq. (7.20) and (7.21)] and also [23]):

- (1) ∇^V is **torsion-free**, namely,

$$\nabla_X^V Y - \nabla_Y^V X = [X, Y]$$

for every vector field X and Y on Ω ,

(2) ∇^V is **almost g -compatible**, namely,

$$X \cdot g_V(Y, Z) = g_V(\nabla_X^V Y, Z) + g_V(Y, \nabla_X^V Z) + 2C_V(\nabla_X^V V, Y, Z),$$

where X, Y , and Z are vector fields on open set Ω and g_V and C_V are the tensors on Ω such that $(g_V)_p = g_{V_p}$ and $(C_V)_p = C_{V_p}$.

It can be checked that the Christoffel symbols of ∇^V only depend on V_p at every $p \in M$, and not on the particular extension of V . Therefore, the Chern connection is an anisotropic connection. Moreover, it is positively homogeneous of degree zero, namely, $\nabla^{\lambda v} = \nabla^v$ for all $v \in TM \setminus \{0\}$ and $\lambda > 0$. For an explicit expression of the Christoffel symbols of the Chern connection in terms of the coefficients of the fundamental and the Cartan tensors, see [9, Eq. (2.4.9)]. The Chern connection provides an (anisotropic) curvature tensor R , which, for every $p \in M$ and $v \in T_p M \setminus \{0\}$, determines a linear map

$$R_v : T_p M \times T_p M \times T_p M \rightarrow T_p M.$$

The Chern curvature tensor can be defined using the anisotropic calculus as in Equation (5.2), or in coordinates, with the help of the non-linear connection (see [10, Eq. (3.3.2)]). Given a smooth curve $\gamma : I \subset \mathbb{R} \rightarrow M$ and a smooth vector field $W \in \mathfrak{X}(\gamma)$ along γ without singularities, where $\mathfrak{X}(\gamma)$ denotes the smooth sections of the pullback bundle $\gamma^*(TM)$ over I , the Chern connection induces a **covariant derivative** D_γ^W along γ , such that

$$(D_\gamma^W X)(t) = \nabla_{\dot{\gamma}(t)}^{W(t)} X,$$

with the identification $X(t) = X_{\gamma(t)}$, when $X \in \mathfrak{X}(M)$. If the image of the curve is contained in a chart (Ω, φ) (using the Christoffel symbols introduced in Equation (5.1)) the covariant derivative is expressed as

$$(D_\gamma^W X)(t) = \sum_{k=1}^n \left(\dot{X}^k(t) + \sum_{i,j=1}^n X^i(t) \dot{\gamma}^j(t) \Gamma_{ij}^k(W(t)) \right) \partial_k,$$

where $X, W \in \mathfrak{X}(\gamma)$ and $X^l(t)$ and $\dot{\gamma}^l(t)$ are the coordinates of $X(t)$ and $\dot{\gamma}(t)$, respectively, for $t \in I$. When $\dot{\gamma}(t) \neq 0$ for all $t \in I$, we can take as a reference vector $W = \dot{\gamma}$. In such a case, we will use the notation $X' := D_\gamma^{\dot{\gamma}} X$ whenever there is no possible confusion about γ .

2.3. GEODESICS AND JACOBI FIELDS. We will say that a smooth curve $\gamma : I \subset \mathbb{R} \rightarrow M$ is a geodesic of (M, F) if it is an auto-parallel curve of the covariant derivative induced by the Chern connection, namely, $D_{\dot{\gamma}}\dot{\gamma} = 0$. Given a vector $v \in TM \setminus \{0\}$, there is a unique geodesic γ_v such that $\dot{\gamma}_v(0) = v$.

When we consider a geodesic variation of γ , it turns out that the variational vector field J is characterized by solving the differential equation

$$(2.1) \quad J''(t) + R_{\dot{\gamma}(t)}(J(t)) = 0,$$

where, for every $p \in M$ and $v \in T_p M \setminus \{0\}$, we define the operator $R_v : T_p M \rightarrow T_p M$ as

$$R_v(w) = R_v(w, v)v$$

for every $w \in T_p M$ (see, for example, [25, Prop. 2.11]). Given a geodesic γ , the operator $R_{\dot{\gamma}}$, defined for vector fields along γ , and the solutions of Equation (2.1) are known, respectively, as the **Jacobi operator** of γ , and the **Jacobi fields** of γ .

When we consider a submanifold L , the minimizers from L are orthogonal geodesics (see [8]), and variations of orthogonal geodesics will be given by L -Jacobi fields, which will be introduced below. First let us define orthogonal vectors. Given a submanifold $L \subset M$, we say that a vector $v \in T_p M$, with $p \in L$, is orthogonal to L if $g_v(v, u) = 0$ for all $u \in T_p L$. The subset of orthogonal vectors to L at $p \in L$, denoted by $\nu_p L$, could be non-linear, but it is a smooth cone (see [27, Lemma 3.3]). We say that a geodesic $\gamma : I \rightarrow M$ is orthogonal to L at $t_0 \in I$ if $\dot{\gamma}(t_0)$ is orthogonal to L .

Definition 2.1: Let L be a submanifold of a Finsler manifold (M, F) and $\gamma : [a, b] \rightarrow M$ a geodesic orthogonal to L at $p = \gamma(a)$. We say that a Jacobi field is **L -Jacobi** if

- $J(a)$ is tangent to L ,
- $\mathcal{S}_{\dot{\gamma}(a)}J(a) = \tan_{\dot{\gamma}(a)} J'(a)$,

where $\mathcal{S}_{\dot{\gamma}} : T_p L \rightarrow T_p L$ is the **shape operator** defined as

$$\mathcal{S}_{\dot{\gamma}}(u) = \tan_{\dot{\gamma}(a)} \nabla_u^{\dot{\gamma}(a)} \xi,$$

with ξ an orthogonal vector field along L such that $\xi_p = \dot{\gamma}(a)$ and $\tan_{\dot{\gamma}(a)}$, the $g_{\dot{\gamma}(a)}$ -orthogonal projection into $T_p L$. An instant t_1 is called **L -focal** (and $\gamma(t_1)$ a **focal point**) if there exists an L -Jacobi field J such that $J(t_1) = 0$.

One can check that the shape operator is symmetric since the connection is symmetric (recall [27, Eq. (16)]). Observe that to compute $\tan_{\gamma(a)} \nabla_u^{\gamma(a)} \xi$, one formally needs a vector field ξ along L which extends ξ_p , but it turns out that this quantity does not depend on the chosen extension, but only on ξ_p (see [27, Prop. 3.5]).

2.4. GEODESIC VECTOR FIELDS. As we have stressed in §2.1, all the computations at a direction $v \in T_p M \setminus \{0\}$ can be made using a vector field V without singularities defined on some neighborhood Ω of $p \in M$ with the affine connection ∇^V . The choice of this V is arbitrary, so, for example, it is possible to choose V such that $\nabla_u^v V = 0$ for all $u \in T_p M$ (see [25, Prop. 2.13]). As, in this work, we will deal mainly with geodesics, it will be particularly convenient to choose a geodesic vector field V . In such a case, it is possible to relate some elements of the Chern connection with the Levi-Civita connection of the Riemannian metric g_V .

PROPOSITION 2.2: *Let V be a geodesic field on an open subset $U \subset M$ and $\hat{g} := g_V$ denote the Riemannian metric on Ω induced by the fundamental tensor g , and let $\hat{\nabla}$ and \hat{R} be the Levi-Civita connection and the Jacobi operator of \hat{g} , respectively. Then, for any $X \in \mathfrak{X}(\Omega)$,*

- (i) $\hat{\nabla}_X V = \nabla_X^V V$ and $\hat{\nabla}_V X = \nabla_V^V X$,
- (ii) $\hat{R}_V X = R_V X$.

As a consequence, the integral curves of V are also geodesics of \hat{g} , and the Finslerian Jacobi operator and Jacobi fields along the integral curves of V coincide with those of \hat{g} .

A proof of this result can be found in [39, Prop. 6.2.2] and also in [25, Prop. 3.9].

2.5. FINSLER SUBMERSIONS AND FOLIATIONS. One of the first examples of Finsler foliations is the partition of M into the fibers of a Finsler submersion. Recall that a submersion $\pi : (M, F) \rightarrow (B, \tilde{F})$ between Finsler manifolds is a **Finsler submersion** if $d\pi_p(B_p^F(0, 1)) = B_{\pi(p)}^{\tilde{F}}(0, 1)$, for every $p \in M$, where $B_p^F(0, 1)$ and $B_{\pi(p)}^{\tilde{F}}(0, 1)$ are the unit balls of the Minkowski spaces $(T_p M, F_p)$ and $(T_{\pi(p)} B, \tilde{F}_{\pi(p)})$ centered at 0, respectively. Examples and a few properties of Finsler submersions can be found in [3] and [7].

Recall that we say that a geodesic $\gamma : I \subset \mathbb{R} \rightarrow M$ is horizontal if $\dot{\gamma}(t)$ is an orthogonal vector to the fiber $\pi^{-1}(\pi(\gamma(t)))$ for every $t \in I$. Here we just need to recall the lift property (see [7, Theorem 3.1]).

PROPOSITION 2.3: *Let $\pi : (M, F) \rightarrow (B, \tilde{F})$ be a Finsler submersion. Then an immersed curve on B is a geodesic if and only if its horizontal lifts are geodesics on M . In particular, the geodesics of (B, \tilde{F}) are precisely the projections of horizontal geodesics of (M, F) .*

This result allows us to get a geodesic field on (M, F) that projects on a geodesic field on (B, \tilde{F}) , which will simplify many computations.

PROPOSITION 2.4: *Let $\pi : (M, F) \rightarrow (B, \tilde{F})$ be a Finsler submersion and V^* a geodesic field in some open subset \tilde{U} of B . Then the horizontal lift V of V^* is a geodesic vector field on $U = \pi^{-1}(\tilde{U})$ and the restriction $\pi|_U : (U, g_V^F) \rightarrow (\tilde{U}, g_{V^*}^{\tilde{F}})$ is a Riemannian submersion, where g^F and $g^{\tilde{F}}$ are the fundamental tensors of F and \tilde{F} , respectively.*

Proof. That V is a geodesic field follows from Proposition 2.3, while the last statement is a consequence of [7, Prop. 2.2]. ■

A **Finsler foliation** of M is a regular foliation \mathcal{F} such that at every point $p \in M$, there exists a neighborhood U in such a way that $\mathcal{F}|_U$ can be obtained as the fibers of a Finsler submersion. This is equivalent to the property of **transnormality**:

$$(2.2) \quad \begin{array}{l} \text{if a geodesic is orthogonal to one leaf of } \mathcal{F}, \\ \text{then it is orthogonal to all the fibers it meets.} \end{array}$$

Recall that a **singular foliation** of M is a partition of M by submanifolds, which are called leaves as in the case of regular foliations, with the following property: given $p \in M$, if $v \in T_p M$ is a vector tangent to a leaf, there exists a smooth vector field X in a neighborhood of p such that $X_p = v$ and X is always tangent to the leaves of the foliation. This is equivalent to saying that for each p there exists a neighborhood U of p , a neighborhood $B(0)$ of 0 in \mathbb{R}^k , and a submersion $\pi : U \rightarrow B(0) \subset \mathbb{R}^k$ so that each fiber $\pi^{-1}(x)$, with $x \in B(0)$, is contained in a leaf, and $P_p = \pi^{-1}(0)$ is a precompact open subset of the leaf L_p that contains p (see [3, Lemma 3.8]). The submanifold P_p is called **plaque**. Note that k here is the codimension of L_p and $B(0)$ can be considered to be a transverse submanifold S_p to P_p . By definition, the

leaves \mathcal{F} on U must intersect S_p . Finally, we say that the singular foliation is **Finsler** if the property (2.2) holds. If P_q is a plaque, there exist future and past tubular neighborhoods denoted, respectively, by $\mathcal{O}(P_q, \varepsilon)$ and $\tilde{\mathcal{O}}(P_q, \varepsilon)$ of a certain radius $\varepsilon > 0$ (see [8]) and in these tubular neighborhoods a singular Finsler foliation can be characterized using the cylinders

$$\mathcal{C}_{r_1}^+(P_q) = \{p \in M : d_F(P_q, p) = r_1\} \quad \text{and} \quad \mathcal{C}_{r_2}^-(P_q) = \{p \in M : d_F(p, P_q) = r_2\},$$

where $d_F(P_q, p)$ is computed as the infimum of the lengths of curves from P_q to p and $d_F(p, P_q)$ as the infimum of the lengths of curves from p to P_q . Recall that given $x \in \mathcal{O}(P_q, \varepsilon)$ (resp. $\tilde{\mathcal{O}}(P_q, \varepsilon)$), the plaque P_x is the connected component which contains x of the intersection of the leaf through x with $\mathcal{O}(P_q, \varepsilon)$ (resp. $\tilde{\mathcal{O}}(P_q, \varepsilon)$).

LEMMA 2.5: *A singular foliation \mathcal{F} is Finsler if and only if its leaves are locally equidistant, i.e., if its leaves satisfy the following property: if a point x belongs to the future cylinder $\mathcal{C}_{r_1}^+(P_q)$ (resp. the past cylinder $\mathcal{C}_{r_2}^-(P_q)$), then the plaque P_x of the future (resp. past) tubular neighborhood is contained in $\mathcal{C}_{r_1}^+(P_q)$ (resp. $\mathcal{C}_{r_2}^-(P_q)$).*

A proof of the above characterization can be found in [3, Lemma 3.7].

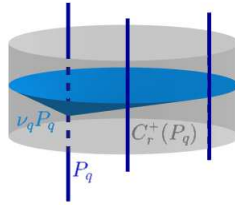


Figure 1. Figure generated by the software geogebra.org, illustrating a few concepts of §2.5, such as a future cylinder $\mathcal{C}_r^+(P_q)$ of a plaque P_q , other plaques contained in this future cylinder and the normal cone at q denoted by $\nu_q P_q$, i.e, the set of vectors $\xi \in T_q M$ so that $g_\xi(\xi, X) = 0$ for all $X \in T_q P_q$. Here the leaves are the fibers of the Finsler submersion $\pi : (M, F) \rightarrow (B, \tilde{F})$, where $M = \mathbb{R}^3$, $B = \mathbb{R}^2$, $\pi(x) = (x_1, x_2)$ and F and \tilde{F} are Randers metrics having the Euclidean metrics and the winds $W = (\frac{1}{2}, 0, \frac{\sin^2(x_1)+1}{4})$ and $\tilde{W} = (\frac{1}{2}, 0)$ as Zermelo data, respectively.

Remark 2.6: Given a (singular) Riemannian foliation \mathcal{F} with closed leaves on a complete Riemannian manifold (M, \mathbf{h}) and an \mathcal{F} -basic vector field W , then \mathcal{F} is a (singular) Finsler foliation for the Randers metric with Zermelo data (\mathbf{h}, W) ; cf. [3, Example 2.13]. The converse is also true. In fact, according with [3, Theorem 1.1], every singular Finsler foliation with closed leaves on Randers spaces (M, F) is produced in this way.

3. Wilking's construction

In [46], Wilking proved the smoothness of the Sharafutdinov retraction and studied dual foliations of singular Riemannian foliations (SRF for short). For that, he used results on self-adjoint spaces, regular distributions along geodesics and Morse–Sturm systems along these distributions; see also [18]. These tools turn out to be quite useful in the study of the transversal geometry of SRF's (see, e.g., [31]). Roughly speaking, given an SRF \mathcal{F} on M , for each horizontal geodesic γ (that may cross singular leaves) one can define a distribution $t \rightarrow \mathcal{H}(t)$ along γ so that $\mathcal{H}(t)$ is the normal distribution $\nu_{\gamma(t)}(L)$ when $\gamma(t)$ lies in a regular leaf; for the sake of simplicity we call this distribution Wilking's distribution. In addition, when the leaves of \mathcal{F} are closed and $\pi : M \rightarrow M/\mathcal{F}$ is the canonical projection, being the quotient M/\mathcal{F} a manifold with the induced Riemannian metric, one can identify the Jacobi fields along $\pi \circ \gamma$ with the solutions of a Morse–Sturm system along $t \rightarrow \mathcal{H}(t)$, the so-called transversal Jacobi field equation.

In this section we review the Finsler version of these objects; see Definition 3.7. We also present a few results on Finsler Jacobi fields, whose proofs will be given in §5.2.

Definition 3.1: Let $\gamma : I \subseteq \mathbb{R} \rightarrow M$ be a geodesic of a Finsler manifold (M, F) and \mathbb{W} a linear subspace of Jacobi fields so that γ is orthogonal to the elements of \mathbb{W} , i.e., $g_{\dot{\gamma}}(\dot{\gamma}, J) = 0$ for every $J \in \mathbb{W}$. The vector space \mathbb{W} is said to be **self-adjoint** if $g_{\dot{\gamma}}(J'_1, J_2) = g_{\dot{\gamma}}(J_1, J'_2)$, for $J_1, J_2 \in \mathbb{W}$.

Remark 3.2: One can prove that given two Jacobi fields J_1 and J_2 along γ , then $g_{\dot{\gamma}}(J'_1, J_2) - g_{\dot{\gamma}}(J_1, J'_2)$ is constant on the interval I . Therefore to show that a vector space \mathbb{W} of Jacobi fields is self-adjoint, it suffices to check that $g_{\dot{\gamma}(t_0)}(J'_1(t_0), J_2(t_0)) = g_{\dot{\gamma}(t_0)}(J_1(t_0), J'_2(t_0))$ for some $t_0 \in I$ (see for, example, [27, Prop. 3.18]).

In the next lemma, we present the distribution $t \rightarrow \mathcal{H}(t)$ associated with a general subspace \mathbf{V} of \mathbf{W} as well as the (generalized) transversal Jacobi field equation.

LEMMA 3.3: *Let \mathbf{W} be an $(n-1)$ -self-adjoint vector space of Jacobi fields orthogonal to a geodesic $\gamma : I \subseteq \mathbb{R} \rightarrow M$ on a Finsler manifold (M^n, F) . Let \mathbf{V} be a linear subspace of \mathbf{W} and hence also a self-adjoint space. Define the subspace $\mathcal{V}(t)$ of $T_{\gamma(t)}M$ for every $t \in I$ as*

$$\mathcal{V}(t) := \{J(t) | J \in \mathbf{V}\} \oplus \{J'(t) | J \in \mathbf{V}, J(t) = 0\}.$$

- (a) *Then $\dim \mathcal{V}(t) = \dim \mathbf{V}$ for every $t \in I$. Furthermore, the second summand is trivial for almost every t .*
- (b) *Let $t \rightarrow \mathcal{H}(t)$ be the orthogonal complement of $\mathcal{V}(t)$ with respect to $g_{\dot{\gamma}(t)}$, i.e., $w \in \mathcal{H}(t)$ if and only if $g_{\dot{\gamma}(t)}(u, w) = 0$ for all $u \in \mathcal{V}(t)$. Let $(\cdot)^{\mathfrak{h}}$ and $(\cdot)^{\mathfrak{v}}$ be the orthogonal projections (with respect to $g_{\dot{\gamma}}$) into $\mathcal{H}(t)$ and $\mathcal{V}(t)$, respectively. Then if $J \in \mathbf{W}$, $J^{\mathfrak{h}}$ fulfills the transversal Jacobi equation, i.e.,*

$$(3.1) \quad (D^{\mathfrak{h}})^2(J^{\mathfrak{h}}) + (R_{\dot{\gamma}}(J^{\mathfrak{h}}))^{\mathfrak{h}} - 3(\mathbb{A}_{\gamma})^2(J^{\mathfrak{h}}) = 0,$$

where $D^{\mathfrak{h}}$ is the induced connection on the horizontal bundle, which is defined as $D^{\mathfrak{h}}(X) = ((X^{\mathfrak{h}})')^{\mathfrak{h}}$, and \mathbb{A}_{γ} is the O' Neill tensor along the geodesic γ , i.e., $\mathbb{A}_{\gamma}(X) := ((X^{\mathfrak{h}})')^{\mathfrak{v}} + ((X^{\mathfrak{v}})')^{\mathfrak{h}}$ for every $X \in \mathfrak{X}(\gamma)$.

- (c) *If J_1, \dots, J_r is a basis of \mathbf{V} with $r = \dim \mathbf{V}$ such that at $t_0 \in I$, $J_k(t_0) = \dots = J_r(t_0) = 0$ and $J_i(t_0) \neq 0$ for $i = 1, \dots, k-1$, then*

$$J_1(t), \dots, J_{k-1}(t), \frac{1}{t-t_0}J_k(t), \dots, \frac{1}{t-t_0}J_r(t)$$

is a continuous basis of $\mathcal{V}(t)$ for every t in $(t_0 - \varepsilon, t_0 + \varepsilon) \cap I \setminus \{t_0\}$ for some $\varepsilon > 0$, with $\lim_{t \rightarrow t_0} \frac{1}{t-t_0}J_i(t) = J'_i(t_0)$ for all $i = k, \dots, r$.

Proof. First, observe that given $t_0 \in I$, there exists a neighborhood $U \subset M$ of $\gamma(t_0)$ which admits a geodesic vector field V such that $\gamma(t) \subset U$ and $V_{\gamma(t)} = \dot{\gamma}(t)$ for all $t \in I_{\varepsilon} = (t_0 - \varepsilon, t_0 + \varepsilon) \cap I$ for some $\varepsilon > 0$. From Proposition 2.2, we conclude that \mathbf{W} and \mathbf{V} are also self-adjoint spaces of (Riemannian) Jacobi fields with respect to the Riemannian metric $\hat{g} := g_V$. Since the lemma is already true for self-adjoint spaces of Riemannian Jacobi fields (see [18, Chapter 1]), it is also valid in the Finsler case for $t \in I$. Indeed, part (a) follows from [18, Lemma 1.7.1] (for the triviality of the second summand, observe that the interval I can be

covered by a countable number of intervals I_{ε_i} with geodesic vector fields V_i , $i \in \mathbb{N}$, as above). Part (b) follows from the equation above [18, Eq. (1.7.7)] observing that if $A(t) : \mathcal{V}(t) \rightarrow \mathcal{H}(t)$ and its dual $A^*(t) : \mathcal{H}(t) \rightarrow \mathcal{V}(t)$ are the linear operators defined below [18, Eq. (1.7.4)] (which are also naturally defined for Finsler metrics), then $\mathbb{A}_\gamma|_{\mathcal{H}} = -A^*$ and $\mathbb{A}_\gamma|_{\mathcal{V}} = A$. In fact, when t_0 is regular (the second summand in part (a) is trivial), for every $u \in \mathcal{V}(t)$, there exists a Jacobi field $J \in \mathcal{V}$ such that $J(t_0) = u$. Then $A(t_0)u = (J')^b(t_0)$, but as $J^\flat = J$, it follows that $\mathbb{A}_\gamma|_{\mathcal{V}(t_0)} = A(t_0)$. Moreover,

$$\begin{aligned} g_\gamma(\mathbb{A}_\gamma|_{\mathcal{H}(t_0)}(X), Z) &= -g_\gamma(X, \mathbb{A}_\gamma|_{\mathcal{V}(t_0)}(Z)) \\ &= g_\gamma(X, -A(t_0)(Z)) = g_\gamma(-A^*(t_0)(X), Z), \end{aligned}$$

which allows us to conclude that $\mathbb{A}_\gamma|_{\mathcal{H}(t_0)} = -A^*(t_0)$. This implies, taking into account the equation above [18, Eq. (1.7.7)], that Equation (3.1) holds for every regular instant, and, by continuity, for all $t \in I$, as regular instants are dense by part (a). We stress that, by Proposition 2.2, the O'Neill tensor of γ computed with g_V coincides with that computed with F . Finally, for part (c), see the proof of [18, Lemma 1.7.1]. ■

As we will see below, an example of $(n-1)$ -self-adjoint space \mathbb{W} is the space of L -Jacobi fields for a submanifold L along a geodesic γ which is orthogonal to L . In the case where L is a fiber of a Finsler submersion $\pi : M \rightarrow B$, \mathcal{V} is the space of holonomic Jacobi fields, i.e., those Jacobi fields whose π -projections are zero. Also in this case, the transversal Jacobi field equation along γ will be identified with a Jacobi field equation along a geodesic in B (see Remark 3.9).

LEMMA 3.4: *Let L be a submanifold on a Finsler manifold (M^n, F) and $\gamma : I \subset \mathbb{R} \rightarrow M$ a geodesic orthogonal to L at $t_0 \in I$. Consider \mathbb{W} the vector space of L -Jacobi fields orthogonal to γ . Then \mathbb{W} is a self-adjoint space of dimension $n-1$.*

Proof. See §5.2.1. ■

PROPOSITION 3.5: *Let (M, F) be a Finsler manifold and L a submanifold of M . Given a geodesic $\gamma : I = [a, b] \subset \mathbb{R} \rightarrow M$ orthogonal to L at the instant t_0 , we have that a vector field J along γ is L -Jacobi if and only if it is the variation vector field of a variation whose longitudinal curves are L -orthogonal geodesics.*

Proof. See §5.2.2. ■

When L is a fiber of a Finsler submersion and J is an L -Jacobi field tangent to the fibers, then J turns out to be the variation vector field of a variation determined by endpoint maps η_ξ^r , as we will see in the next lemma.

LEMMA 3.6: Consider a Finsler submersion $\pi : (M, F) \rightarrow (B, \tilde{F})$, and set $L = \pi^{-1}(b)$ for some $b \in B$. Let $\gamma : I \rightarrow M$ be a geodesic orthogonal to L at $t_0 \in I$ and $t \rightarrow J(t)$ be an L -Jacobi field along γ with $J(t_0)$ tangent to $T_p L$ where $p = \gamma(t_0)$. Also assume that J is g_γ -orthogonal to γ . Then the following items are equivalent:

- (a) J is a vertical (holonomy) Jacobi field, i.e., J is always tangent to the fibers of π . In other words $J^\flat = 0$.
- (b) There exists a curve $\beta : (-\varepsilon, \varepsilon) \subset \mathbb{R} \rightarrow L$ with $J(t_0) = \dot{\beta}(0)$, such that if ξ is the normal basic vector field along L with $\xi_p = \dot{\gamma}(t_0)$ and $\psi : I \times (-\varepsilon, \varepsilon) \rightarrow M$ is the variation

$$(t, s) \rightarrow \psi(t, s) = \gamma_{\xi_{\beta(s)}}(t - t_0) = \eta_\xi^{t-t_0}(\beta(s)),$$

then

$$J(t) = \frac{\partial}{\partial s} \psi(t, 0).$$

Proof. See §5.2.3. ■

Definition 3.7: Let \mathcal{F} be a partition by stratified submanifolds of M such that there exists an open neighborhood $U \subset M$ where the partition $\mathcal{F}|_U$ is a regular Finsler foliation of U . Given a geodesic $\gamma : I \subseteq \mathbb{R} \rightarrow M$ and $t_0 \in I$ such that $\gamma(t_0) \in U$ and $\dot{\gamma}(t_0) \in \nu(L_{\gamma(t_0)})$, denoting by L_p the fiber of $\mathcal{F}|_U$ which contains $p \in U$, let \mathbb{W} be the self-adjoint space of $L_{\gamma(t_0)}$ -Jacobi fields orthogonal to γ defined in Lemma 3.4 and \mathbb{V} the subspace of \mathbb{W} such that $J \in \mathbb{V}$ iff $J(t) \in T_{\gamma(t)} L_{\gamma(t)}$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, where $\varepsilon > 0$ is small enough to guarantee that $\gamma(t_0 - \varepsilon, t_0 + \varepsilon) \subset U$. Then the distributions $t \rightarrow \mathcal{V}(t)$, $t \rightarrow \mathcal{H}(t)$ and the differential equation (3.1) associated with the self-adjoint spaces \mathbb{W} and \mathbb{V} are called the **Wiling distributions** and the **transversal Jacobi field equation** along γ starting at $\{\gamma(t_0), L_{\gamma(t_0)}\}$.

Remark 3.8: Note that even when \mathcal{F} is a singular foliation, the distributions $t \rightarrow \mathcal{V}(t)$, $t \rightarrow \mathcal{H}(t)$ and the differential equation (3.1) are well defined for all t including the case where $\gamma(t)$ is a singular point. Also note that the transversal

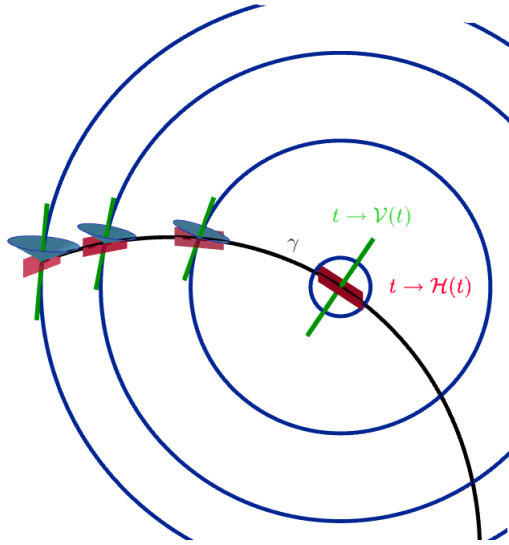


Figure 2. Figure generated by the software geogebra.org, illustrating the Wilking distributions $t \rightarrow \mathcal{H}(t)$ and $t \rightarrow \mathcal{V}(t)$ associated with the singular Finsler foliation $\mathcal{F}_\pi = \{\pi^{-1}(c)\}$ on the neighborhood $B_{\frac{3}{2}}(0) \subset \mathbb{R}^3$ where $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is defined as $\pi(x) = (x_1^2 + x_2^2, x_3)$. The Finsler metric F restricted to $B_{3/2}(0)$ is the Randers metric having the Euclidean metric and the wind $W = (-\frac{x_2}{2}, \frac{x_1}{2}, 0)$ as Zermelo data.

Jacobi field equation (3.1) is a Morse–Sturm system; see [17, 29, 31, 37]. Although the distribution $\mathcal{V}(t)$ coincides with the tangent space of $L_{\gamma(t)}$ until γ crosses a singular leaf, it does not follow straightforwardly that $\mathcal{V}(t)$ should always be tangent to the regular leaves for all t (even at regular leaves after crossing the first singular leaf). This is the case when \mathcal{F} is the foliation studied in §4 obtained as the fibers of an analytic Finsler submersion.

Remark 3.9: By applying Proposition 3.5 we can infer useful natural interpretations of the transversal Jacobi field equation. Let $\pi : M \rightarrow B$ be a Finsler submersion and L a fiber, i.e., $L = \pi^{-1}(b)$, for any $b \in B$. Let $p \in L$ and γ_p be a horizontal geodesic starting at p , i.e., $p = \gamma_p(0)$. Then the transversal Jacobi equation along γ_p starting at $L_{\gamma_p(0)}$ can be identified with the Jacobi

field equation along the geodesic $\pi \circ \gamma_p$ in B . Indeed, the transversal Jacobi equation is the horizontal lift to M of the Jacobi equation on B . To check this, consider a geodesic vector field V whose integral curves are horizontal geodesics of (M, F) , being $V^* = d\pi \circ V$, and recall that, by Proposition 2.4, $\pi : (M, g_V) \rightarrow (B, \tilde{g}_{V^*})$ is a Riemannian submersion. Then use the relation between the Levi-Civita connection of (B, \tilde{g}_{V^*}) and the horizontal part of the Levi-Civita connection of (M, g_V) when applied to basic vector fields (see [34, Lemma 1 (3)]) and the relation between the horizontal part of their curvature tensors in [34, Theorem 2 {4}] taking into account Proposition 2.2. In addition, if $x \in L$ and γ_x is a horizontal geodesic starting at x so that $\pi \circ \gamma_x = \pi \circ \gamma_p$, then the transversal Jacobi field equation along γ_p starting at L_p and the transversal Jacobi field equation along γ_x starting at L_x can be identified with each other. As we will see from Lemmas 4.3 and 4.5, this interpretation will also hold for analytic singular Finsler submersions.

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1 adapting an argument of [1] and using Wilking's distributions; recall Definition 3.7. Let $\pi : M^n \rightarrow B^k$ be an analytic map. In the first subsection, assuming π is a Finsler submersion on an open subset of M , we prove that π is a Finsler submersion on the regular part M_0 and the regular fibers are equifocal submanifolds. In the second part, we will assume that the fibers constitute a singular foliation and will prove the transnormality.

4.1. EQUIFOCALITY. Let $q \in M$ be a regular point of π , $c = \pi(q)$ and $L = \pi^{-1}(c)$. We will consider a unit basic vector field ξ along a neighborhood of q in L , i.e., ξ is orthogonal to L , projectible and $F(\xi) = 1$. We will show that L is a regular fiber of π by extending this basic vector field along L (Lemma 4.1) and will check part (b) of Theorem 1.1. More precisely, in Lemma 4.3, we will prove that $\eta_\xi^r(L)$ is contained in a fiber of π and in Lemmas 4.9 and 4.10, that $d\eta_\xi^r$ has constant rank for all r .

LEMMA 4.1: *If $p \in M$ is a regular point of π and there exists a neighborhood U of p in (the regular part of) L_p such that F is constant on basic vector fields on U , then L_p is regular and F is constant on basic vector fields on L_p .*

Proof. Let us denote by L_p^R the connected component of p of this regular part (which is open) and ξ a basic vector field on L_p^R , which is analytic as all the data is analytic. Then as, by hypothesis, the function $F \circ \xi : L_p^R \rightarrow \mathbb{R}$ is constant in a neighborhood of p , it must be constant everywhere. Now consider a point q in the boundary of L_p^R . As $F \circ \xi$ is constant, if one considers a sequence of points $\{q_n\} \subset L_p^R$ such that $\lim_{n \rightarrow \infty} q_n = q$, then the sequence of vectors ξ_{q_n} is precompact in TM and it converges to some $v_q \in T_q M$ up to a subsequence. Let $\tilde{\xi} = d\pi(\xi)$, which is well-defined because ξ is basic. By continuity, the vector v_q projects to $\tilde{\xi}$, and since this can be done for every $\tilde{\xi} \in T_{\pi(p)} B$ (with the corresponding basic lift), it turns out that the points of the boundary are regular and F is constant on ξ . \blacksquare

Remark 4.2: As a consequence of Lemma 4.1, the assumption that \mathcal{F}_π is a Finsler foliation restricted to some open subset $U \subset M$ implies that there exists a saturated open subset where π is a Finsler submersion. Namely, consider $\pi(U)$, which is open because π is a submersion and then $\pi|_{\pi^{-1}(\pi(U))}$ is a Finsler submersion for a certain Finsler metric on $\pi(U)$.

LEMMA 4.3: *For all $r \in \mathbb{R}$, there exists a value $d \in B$ such that $\eta_\xi^r(L) \subset \pi^{-1}(d)$.*

Proof. Assume that $r > 0$ (the case $r < 0$ is analogous and $r = 0$ is trivial). Consider $x, y \in L$ and set

$$\tilde{I} := \{t_0 \in [0, r] : \pi(\gamma_{\xi_x}(t)) = \pi(\gamma_{\xi_y}(t)) \text{ for } t \leq t_0\}.$$

Note that the set \tilde{I} is not empty, because π restricted to a neighborhood of L is a Finsler submersion (recall Remark 4.2). The set \tilde{I} is closed due to continuity and it is an open set because π is an analytic map and geodesics of an analytic Finsler metric are also analytic. Therefore $\tilde{I} = [0, r]$ and this concludes the proof. \blacksquare

Fix $x \in L$, and consider the geodesic γ_{ξ_x} . Since, by Remark 4.2, π is a Finsler submersion on a neighborhood of x , we can define along the geodesic γ_{ξ_x} the Wilking distribution pair $t \rightarrow (\mathcal{H}(t), \mathcal{V}(t))$ (recall Definition 3.7).

LEMMA 4.4: *If $x \in L$ and $p = \gamma_{\xi_x}(t_0)$ is a regular point of π , then the Wilking distribution $\mathcal{V}(t_0)$ coincides with the tangent space to L_p . Moreover, the distributions $t \rightarrow \mathcal{V}(t)$ and $t \rightarrow \mathcal{H}(t)$ are analytic and the singular points of γ_{ξ_x} (lying in a singular level set) are isolated.*

Proof. Recall by part (c) of Lemma 3.3 that there is a basis J_1, \dots, J_r of \mathbf{V} with $r = \dim \mathbf{V}$ such that at t_0 , $J_k(t_0) = \dots = J_r(t_0) = 0$ and $J_i(t_0) \neq 0$ for $i = 1, \dots, k-1$. Then

$$(4.1) \quad J_1(t), \dots, J_{k-1}(t), \frac{1}{t-t_0} J_k(t), \dots, \frac{1}{t-t_0} J_r(t)$$

is a continuous basis of $\mathcal{V}(t)$ for every t in $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$, with $\lim_{t \rightarrow t_0} \frac{1}{t-t_0} J_i(t) = J'_i(t_0)$ for all $i = k, \dots, r$. By Lemma 3.6, $J_i(t) = d\eta_{\xi}^t(v_i)$ for some vector $v_i \in T_x L$ for all $i = 1, \dots, r$, and by Lemma 4.3, all $J_i(t)$ are tangent to $L_{\gamma_{\xi_x}(t)}$ for all t . By continuity, $J'_j(t)$ is also tangent to $L_{\gamma_{\xi_x}(t)}$ for $j = k, \dots, r$. This implies that $\mathcal{V}(t_0)$ is contained in the tangent space to L_p . As they have the same dimension, they coincide. The analyticity of \mathcal{V} and \mathcal{H} follows from the basis of $\mathcal{V}(t)$ in (4.1), as the vector fields $\tilde{J}_j(t) = \frac{1}{t-t_0} J_j(t)$ extend analytically to t_0 making

$$\tilde{J}_j(t_0) = J'_j(t_0)$$

as $J_j(t_0) = 0$ and J_j is analytic, for $j = k, \dots, r$. Up to the singular points of γ_{ξ_x} , consider an analytic frame of the horizontal space \mathcal{H} , namely, analytic vector fields along γ_{ξ_x} , H_1, \dots, H_{n-r} such that $H_1(t), \dots, H_{n-r}(t)$ is a basis of $\mathcal{H}(t)$ for every t (this can be obtained using the D^b -parallel transport). Then when choosing an analytic frame along $\pi \circ \gamma_{\xi_x}$, E_1, \dots, E_{n-r} , the determinant of the transformation matrix with respect to $d\pi \circ H_1, \dots, d\pi \circ H_{n-r}$ can only have isolated zeroes or being zero everywhere. As it is not zero close to t_0 , it has only isolated zeroes. Even if one cannot ensure the existence of a global frame for $\pi \circ \gamma_{\xi_x}$, it is enough to recover its domain with frames which intersect in open subsets to conclude that the singular points along γ_{ξ_x} are isolated. Observe that, by Lemma 4.1, the singular points of π can only lie in singular levels. ■

LEMMA 4.5: *If $p = \gamma_{\xi_x}(t)$ is a regular point of π , then γ_{ξ_x} is orthogonal to L_p .*

Proof. Observe that $f_i(t) = g_{\dot{\gamma}_{\xi_x}}(\dot{\gamma}_{\xi_x}(t), J_i(t))$ and $h_i(t) = g_{\dot{\gamma}_{\xi_x}}(\dot{\gamma}_{\xi_x}(t), J'_i(t))$ are analytic functions for each $J_i \in \mathbf{V}$ in the basis of part (c) of Lemma 3.3, and if $\gamma_{\xi_x}(t)$ is a regular point of π , then, by Lemma 4.4, $T_{\gamma_{\xi_x}(t)} L_{\gamma_{\xi_x}(t)} = \mathcal{V}(t)$. As π is a Finsler submersion in a neighborhood of L (recall Remark 4.2), f_i and h_i are identically zero, and, in particular, by part (c) of Lemma 3.3, γ_{ξ_x} is orthogonal to L_p . ■

LEMMA 4.6: *If there exists a neighborhood U of M such that $\mathcal{F}_\pi|_U$ is a Finsler foliation, then \mathcal{F}_π is a Finsler foliation when restricted to the open subset of regular points of π .*

Proof. Fix $q \in U$. The first observation is that, by Remark 4.2, π is a Finsler submersion when restricted to $\pi^{-1}(\pi(U))$ and L_q is a closed submanifold. Given $p \in M$, a regular point of M , there exists a minimizing geodesic $\gamma: [0, b] \rightarrow M$ from L_q to p , and it follows from the Finsler Morse index Theorem (see [36]) that there is no focal point on $[0, b)$. As γ minimizes the distance from L_q , it must be orthogonal to L_q , and then, by Lemma 4.5, orthogonal to all the regular fibers of π . Now consider the basic vector field ξ along L_q such that $\xi_{\gamma(0)} = \dot{\gamma}(0)$. Then the map $\eta_\xi^t: L_q \rightarrow L_{\gamma(t)}$ is a diffeomorphism on a neighborhood of $\gamma(0)$ for every $t \in [0, b)$. Fix an instant $t_0 \in (0, b)$ such that $\gamma(t_0)$ is a regular point of π (recall that, by Lemma 4.4, singular points are isolated along γ), then by continuity and the compactness of $[0, t_0]$, it is possible to choose a neighborhood $\tilde{U} \subset L_q$ where $\eta_\xi^t: \tilde{U} \rightarrow L_{\gamma(t)}$ is a diffeomorphism onto the image for all $t \in [0, t_0]$. Given a vector v horizontal to $\tilde{U}_0 = \eta_\xi^{t_0}(\tilde{U})$ at $p_0 = \eta_\xi^{t_0}(q_0)$, for some $q_0 \in \tilde{U}$, consider the Wilking distributions $\mathcal{H}^{q_0}(t_0)$ and $\mathcal{V}^{q_0}(t_0)$ along the horizontal curve $\gamma_{\xi_{q_0}}$, and let v^h be the projection of v to $\mathcal{H}^{q_0}(t_0)$. Consider the D^h -parallel vector field X along $\gamma_{\xi_{q_0}}$ such that $X(t_0) = v^h$ and the vector field Y along $\gamma_{\xi_{q_0}}$ orthogonal to $\eta_\xi^t(\tilde{U})$ at each $t \in [0, t_0]$ such that its Wilking projection to \mathcal{H}^{q_0} is X (recall [3, Lemma 2.9 (a)]). Let v_1 be the orthogonal vector to L_q at $\gamma(0)$ such that $d\pi(v_1) = d\pi(Y(0))$. Now repeat the process to obtain a vector field along $\mathcal{H}^{\gamma(0)}$ (the horizontal Wilking distribution along γ) and \tilde{Y} horizontal to $\eta_\xi^t(\tilde{U})$ such that its Wilking projection to $\mathcal{H}^{\gamma(0)}(t)$, $\tilde{X} = \tilde{Y}^h$, is D^h -parallel and $\tilde{X}(0) = v_1^h$ (see Figure 3). Let us observe that if we prove that

$$(4.2) \quad d\pi(Y(t)) = d\pi(\tilde{Y}(t)) \quad \text{for all } t \in [0, t_0],$$

then, for t close to 0, $F(Y(t)) = F(\tilde{Y}(t))$, because π is a Finsler submersion, but, by analyticity, this will be true for all $t \in [0, t_0]$. This implies that $F(v) = F(\tilde{Y}(t_0))$, and, in particular, that F is constant on basic vector fields along \tilde{U}_0 . By Lemma 4.1, F is constant on basic vector fields along the whole fiber $L_{\gamma(t_0)}$. Finally, by continuity, this is also true for L_p . So, let us prove Equation (4.2), which is equivalent to proving that $d\pi(X(t)) = d\pi(\tilde{X}(t))$.

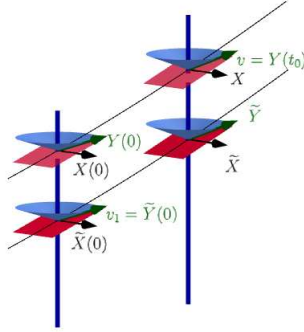


Figure 3. Figure generated by the software geogebra.org, illustrating the definition of the vector fields X and Y along γ_{ξ_0} and \tilde{X} and \tilde{Y} along γ .

This equivalence follows from the fact that

$$d\pi(\mathcal{V}^{q_0}(t)) = d\pi(\mathcal{V}^{\gamma(0)}(t)) = \{0\},$$

because

$$T_{\gamma_{\xi_{q_0}}(t)}\eta_{\xi}^t(\tilde{U}) = \mathcal{V}^{q_0}(t), \quad T_{\gamma(t)}\eta_{\xi}^t(\tilde{U}) = \mathcal{V}^{\gamma(0)}(t)$$

and

$$\eta_{\xi}^t(\tilde{U}) \subset L_{\gamma_{\xi_{q_0}}(t)} = L_{\gamma(t)}$$

(recall that we assume that $\eta_{\xi}^t|_{\tilde{U}}$ is a diffeomorphism onto the image for $t \in [0, t_0]$). By analyticity, we only have to prove that this holds for t close to 0. But this is true, because the D^h -parallel transport in a Riemannian submersion is the horizontal lift of the parallel transport in the base. In our case, the D^h -parallel transport coincides with the lift of the parallel transport induced by \tilde{g}_{V^*} , where V^* is a (local) geodesic vector field in the base tangent to $\pi \circ \gamma$, considering the Riemannian metric g_V on M , where V is the horizontal lift of V^* , being g and \tilde{g} , the fundamental tensors of the Finsler metrics on M and B , respectively. Recall that the normal distribution with respect to g_V coincides with the Wilking distribution and then we can use Proposition 2.4. So, the proof is concluded. ■

Remark 4.7: Observe that as a consequence of Lemmas 4.4, 4.5 and 4.6, given a geodesic which is horizontal in one regular point, then it is horizontal in all the regular points and, by continuity (using Lemma 4.4), the Wilking distribution does not depend on the regular instant chosen as initial point. Let us see that the solutions to the transversal Jacobi equation of two horizontal geodesics with the same projection can be identified.

LEMMA 4.8: *Consider a geodesic γ of (M, F) which is horizontal at the regular points. It holds that*

- (a) *if X is a solution of the transversal Jacobi equation of γ in Equation (3.1), then along the regular instants of γ , $d\pi \circ X$ is a Jacobi field of $\pi \circ \gamma$, and the solutions of the transversal Jacobi equation are characterized by this property,*
- (b) *if α is another geodesic which is horizontal in the regular points with $\pi \circ \gamma = \pi \circ \alpha$, then there exists a vector field Y along α which fulfills the transversal Jacobi equation of α and such that $d\pi \circ X = d\pi \circ Y$ and $g_{\dot{\gamma}}(X, X) = g_{\dot{\alpha}}(Y, Y)$,*
- (c) *γ and α provide the same conjugate points of the transversal Jacobi equation.*

Proof. For part (a), recall Remark 3.9, and use that the singular points are isolated (see Lemma 4.4). For part (b), observe that part (a) implies part (b) when we restrict γ and α to an interval with regular points. Analyticity implies that Y can be extended to the whole interval with the required properties. Part (c) is a straightforward consequence of part (b). ■

Recall that $\pi : M \rightarrow B$ is the analytic map of Theorem 1.1, $L = \pi^{-1}(c)$ is a regular fiber and ξ is a basic vector field along L . Moreover, by Lemma 4.3, for all $r \in \mathbb{R}$, there exists a value $d \in B$ such that $\eta_{\xi}^r(L) \subset \pi^{-1}(d)$.

LEMMA 4.9: *If d is a regular value, then $\eta_{\xi}^r : L \rightarrow \pi^{-1}(d)$ is a diffeomorphism.*

Proof. By Lemma 4.5, $\dot{\gamma}_{\xi_p}(r)$ is orthogonal to $\pi^{-1}(d)$. As π is a Finsler submersion on $\pi^{-1}(d)$ (recall Lemma 4.6), we can extend the normal vector $\dot{\gamma}_{\xi_p}(r)$ to a unit basic vector field $\tilde{\xi}$ along the fiber $\pi^{-1}(d)$. It is also possible to check that $\eta_{\tilde{\xi}}^{-r}$ is the inverse of η_{ξ}^r . ■

We have just proved in Lemma 4.9 that $\eta_\xi^r : \pi^{-1}(c) \rightarrow \pi^{-1}(d)$ is a diffeomorphism when d is a regular value, which implies that $d\eta_\xi^r$ has constant rank. We have to check now that $d\eta_\xi^r$ has constant rank when d is a singular value. This will be done in Lemma 4.10.

LEMMA 4.10: *Consider $\eta_\xi^r : L \rightarrow \pi^{-1}(d)$ with $d \in B$ a singular value. Then $\dim \text{rank } d\eta_\xi^r$ is constant along L .*

Proof. As L is connected, it is enough to prove that $\dim \text{rank } d(\eta_\xi^r)_x$ is locally constant. Consider a point $p \in L$. Since the transversal Jacobi field equation along γ_{ξ_p} starting at L is a Morse–Sturm system, there exists a $\delta > 0$ so that any two points in $I_\delta = (r - \delta, r + \delta)$ are not conjugate to each other. In other words, if X is a solution of the transversal Jacobi field along γ_{ξ_p} , and $s_1, s_2 \in I_\delta$ so that $X(s_1) = 0 = X(s_2)$, then $X(t) = 0$ for all t ; see, e.g., [37, Lemma 2.1]. Also note that this $\delta > 0$ is the same for other γ_{ξ_x} for $x \in L$ as a consequence of part (c) of Lemma 4.8. Using again that the critical points of π on γ_{ξ_x} are isolated, we can also suppose, reducing δ if necessary, that the point $\gamma_{\xi_x}(r)$ is the only critical point of π on $\gamma_{\xi_x}|_{I_\delta}$ for $x \in L$.

Let $\tilde{s} \in (r - \delta, r)$ and $\tilde{p} = \gamma_{\xi_p}(\tilde{s})$. Define the unit vector field $\tilde{\xi}$ along $L_{\tilde{p}}$ so that $\tilde{\xi}_{\tilde{p}} = \dot{\gamma}_{\xi_p}(\tilde{s})$. Recall that $\eta_{\tilde{\xi}}^{\tilde{s}} : L_p \rightarrow L_{\tilde{p}}$ is a diffeomorphism (see Lemma 4.9). Set $\tilde{r} := r - \tilde{s}$. Also note that

$$\eta_\xi^r = \eta_{\tilde{\xi}}^{\tilde{r}} \circ \eta_{\tilde{\xi}}^{\tilde{s}}.$$

Therefore to prove that $x \rightarrow \dim \text{rank } d(\eta_\xi^r)_x$ is locally constant at $p \in L$, it suffices to prove that $\tilde{x} \rightarrow \dim \text{rank } d(\eta_{\tilde{\xi}}^{\tilde{r}})_{\tilde{x}}$ is locally constant at $\tilde{p} = \eta_{\tilde{\xi}}^{\tilde{s}}(p) \in L_{\tilde{p}}$.

We claim that $L_{\tilde{p}}$ -focal points along the horizontal segment of geodesic $\gamma_{\tilde{\xi}_{\tilde{x}}}|_{(0, \tilde{r} + \delta)}$ are of tangential type. In other words, $\gamma_{\tilde{\xi}_{\tilde{x}}}(t_1)$ is a focal point with multiplicity k ($0 < t_1 < \tilde{r} + \delta$) if and only if \tilde{x} is a critical point of the endpoint map $\eta_{\tilde{\xi}}^{t_1}$ and $\dim \ker d\eta_{\tilde{\xi}}^{t_1} = k$.

In order to prove the claim, consider an $L_{\tilde{p}}$ -Jacobi field \tilde{J} along

$$t \rightarrow \gamma_{\tilde{\xi}_{\tilde{x}}}(t) = \gamma_{\xi_x}(t + \tilde{s})$$

and assume that $\tilde{J}(t_1) = 0$ for some $0 < t_1 < \tilde{r} + \delta$. Let J be the Jacobi field along γ_{ξ_x} so that $\tilde{J}(t) = J(t + \tilde{s})$. Observe that J is the variation vector field of a variation by orthogonal geodesics to $L_{\tilde{p}}$ (see Proposition 3.5) and then its projection is a Jacobi field of $\pi \circ \gamma_{\xi_x}$. By part (a) of Lemma 4.8, J^h is a solution of the transversal Jacobi equation. As $J^h(\tilde{s}) = 0 = J^h(\tilde{s} + t_1)$, from

the choice of I_δ and part (c) of Lemma 4.8, we conclude that $J^h(t) = 0$ for all t . Therefore $\tilde{J}^h(t) = 0$ for all t . This fact and Lemma 3.6 imply that $\gamma_{\tilde{\xi}_{\tilde{x}}}(t_1)$ is an $L_{\tilde{p}}$ -focal point if and only if \tilde{x} is a critical point of the endpoint map $\eta_{\tilde{\xi}}^{t_1}$.

From what we have discussed above, we have concluded that

$$(4.3) \quad m(\gamma_{\tilde{\xi}_{\tilde{x}}}) = \dim \ker d\eta_{\tilde{\xi}}^{\tilde{r}}(\tilde{x}),$$

where $m(\gamma_{\tilde{\xi}_{\tilde{x}}})$ denotes the number of $L_{\tilde{p}}$ -focal points on $\gamma_{\tilde{\xi}_{\tilde{x}}} \mid_{(0, \tilde{r}+\delta)}$, each counted with its multiplicities.

On the other hand, observe that

$$(4.4) \quad m(\gamma_{\tilde{\xi}_{\tilde{x}}}) \geq m(\gamma_{\tilde{\xi}_{\tilde{p}}}),$$

for all \tilde{x} in some neighborhood of \tilde{p} in $L_{\tilde{p}}$, as we know that there is a subspace of dimension $m(\gamma_{\tilde{\xi}_{\tilde{p}}})$ where the $L_{\tilde{p}}$ -index form of $\gamma_{\tilde{\xi}_{\tilde{p}}}$ is negative definite, and this will remain true for the $L_{\tilde{p}}$ -index form of $\gamma_{\tilde{\xi}_{\tilde{x}}}$ for \tilde{x} close enough to \tilde{p} .

Equations (4.3) and (4.4) together with the elementary inequality

$$\dim \ker d\eta_{\tilde{\xi}}^{\tilde{r}}(\tilde{x}) \leq \dim \ker d\eta_{\tilde{\xi}}^{\tilde{r}}(\tilde{p})$$

(as $\dim \operatorname{Im} d\eta_{\tilde{\xi}}^{\tilde{r}}(\tilde{x}) \geq \dim \operatorname{Im} d\eta_{\tilde{\xi}}^{\tilde{r}}(\tilde{p})$) imply that $\dim \ker d\eta_{\tilde{\xi}}^{\tilde{r}}$ is constant in a neighborhood of \tilde{p} on $L_{\tilde{p}}$. As $\eta_{\tilde{\xi}}^{\tilde{s}} : L_p \rightarrow L_{\tilde{p}}$ is a diffeomorphism, it follows that $\dim \ker d\eta_{\tilde{\xi}}^{\tilde{r}}$ is constant in a neighborhood of p on L_p , which concludes the proof. \blacksquare

4.2. FINSLERIAN CHARACTER. In this section, we will additionally assume that the fibers constitute a singular foliation and will prove its transnormality. This will follow directly from Lemmas 4.11, 4.5 and 4.15.

LEMMA 4.11: *If \mathcal{F}_π is a singular foliation, γ is a geodesic with $\dot{\gamma}(0) \in \nu(L_{\gamma(0)})$ and $\gamma(0)$ is a regular point, then γ is horizontal.*

Proof. By Lemma 4.5, the geodesic γ is orthogonal to the regular leaves associated with regular values of π . Now suppose that $\gamma(t_0)$ is a singular point and $u \in T_{\gamma(t_0)}L_{\gamma(t_0)}$. Then as (M, \mathcal{F}) is a singular foliation, there exists a vector field X in a neighborhood of $\gamma(t_0)$ such that X is tangent to the leaves and $X_{\gamma(t_0)} = u$. By Lemma 4.4, the singular points along γ are isolated, so $g_{\dot{\gamma}(t)}(X_{\gamma(t)}, \dot{\gamma}(t)) = 0$ for all t close enough to t_0 . By continuity, we conclude that $g_{\dot{\gamma}(t_0)}(u, \dot{\gamma}(t_0)) = 0$, as desired. \blacksquare

Note that the above lemma does not directly imply that the foliation \mathcal{F}_π is a singular Finsler foliation, because the lemma was only proved for a geodesic γ starting at a regular point $p \in L$.

LEMMA 4.12: *If \mathcal{F}_π is a singular foliation and c is a regular value of π , then $\eta_\xi^r : \pi^{-1}(c) \rightarrow \pi^{-1}(d)$ is surjective even when d is a singular value.*

Proof. The proof consists of two steps. In the first step we are going to check that the endpoint map $\eta_\xi^r : \pi^{-1}(c) \rightarrow \pi^{-1}(d)$ is an open map. And in the second step we have to check that the set $\eta_\xi^r(\pi^{-1}(c))$ is a closed subset of $\pi^{-1}(d)$. This ends the proof because the fibers are connected.

For $q = \eta_\xi^r(x_0) \in \pi^{-1}(d)$, consider a plaque P_q of $\pi^{-1}(d)$ which contains q and a past tubular neighborhood $U = \tilde{\mathcal{O}}(P_q, \varepsilon)$ of the plaque P_q of radius $\varepsilon > 0$. By Lemma 4.9 and the fact that the singular points along $\gamma_{\xi_{x_0}}$ are isolated (see Lemma 4.4) and using Lemma 4.11, we can assume without loss of generality that $x_0 \in U$ and $\gamma_{\xi_{x_0}}$ is the minimizing geodesic from x_0 to P_q . This implies that all the geodesics γ_{ξ_x} with $x \in \tilde{U} \subset U \cap \pi^{-1}(c)$, \tilde{U} being a small enough open subset of $\pi^{-1}(c)$ which contains x_0 , are minimizing geodesics from x to P_q , and then

$$(4.5) \quad \eta_\xi^r = \pi^\nu|_{\tilde{U}},$$

where π^ν is the (past) footpoint projection. Since \mathcal{F}_π is a singular foliation, the map $\pi^\nu|_{\tilde{U}}$ is open by [3, Lemma 3.10]. This fact and Equation (4.5) imply that the endpoint map η_ξ^r is an open map.

Consider a sequence $\{y_n\} \subset \eta_\xi^r(\pi^{-1}(c))$ that converges to a point $y \in \pi^{-1}(d)$. Let $\{x_n\} \subset \pi^{-1}(c)$ be a sequence so that $\eta_\xi^r(x_n) = y_n$. Since, for n big enough, $x_n \in \overline{B^-(y, 2r)}$, denoting

$$B^-(y, 2r) = \{x \in M : d_F(x, y) < 2r\},$$

namely, the backward ball of radius $2r$ centered at y , we have a convergent subsequence $x_{n_i} \rightarrow x \in \overline{B^-(y, 2r)}$. Therefore

$$\eta_\xi^r(x) = \lim \eta_\xi^r(x_{n_i}) = \lim y_{n_i} = y$$

and hence $y \in \eta_\xi^r(\pi^{-1}(c))$. ■

LEMMA 4.13: *Let L_p be a regular leaf, and L_q be a singular leaf. Then they are parallel, i.e., if y_0 and y_1 are points of L_q , then*

$$d(L_p, y_0) = d(L_p, y_1) \quad \text{and} \quad d(y_0, L_p) = d(y_1, L_p).$$

Proof. For $i = 0, 1$, let $\gamma_i : [0, r_i] \rightarrow M$ be a minimizing segment of geodesic joining L_p with y_i , i.e., $\gamma_i(0) \in L_p$, $\gamma_i(r_i) = y_i$ and

$$d(L_p, y_i) = \int_0^{r_i} F(\dot{\gamma}_i) dt =: \ell(\gamma_i).$$

Since $\eta_{\xi_1}^{r_1}$ is surjective by Lemma 4.12, we can transport γ_1 and get a (maybe non-unique) new curve $||\gamma_1$ joining L_p to y_0 such that $\ell(||\gamma_1) = \ell(\gamma_1)$. Since γ_0 is minimal, we conclude that $\ell(\gamma_0) \leq \ell(\gamma_1)$. An analogous argument allows us to conclude that $\ell(\gamma_1) \leq \ell(\gamma_0)$ and hence, $\ell(\gamma_1) = \ell(\gamma_0)$ as required, which implies that $d(L_p, y_0) = d(L_p, y_1)$. The other equality is analogous. \blacksquare

LEMMA 4.14: *Given a point \tilde{q} in a singular leaf $L_{\tilde{q}}$, there exists a plaque $P_{\tilde{q}}$ of \tilde{q} admitting a future (resp. past) tubular neighborhood such that if P_q is a (regular or singular) plaque in this neighborhood, then $P_q \subset \mathcal{C}_r^+(P_{\tilde{q}})$ (resp. $P_q \subset \mathcal{C}_r^-(P_{\tilde{q}})$) for an appropriate r .*

Proof. We will consider only the future case, as the past one can be recovered by using the reverse Finsler metric $\tilde{F}(v) = F(-v)$. Let \mathcal{U} be a totally convex neighborhood of \tilde{q} , namely, given two points $p_1, p_2 \in \mathcal{U}$ there exists a unique geodesic joining p_1 to p_2 and this geodesic is minimizing (see [44, 45] for their existence). Now consider a plaque $\tilde{P}_{\tilde{q}}$ of \tilde{q} , a future tubular neighborhood $\tilde{U} \subset \mathcal{U}$, a smaller plaque $P_{\tilde{q}}$ of \tilde{q} and a future tubular neighborhood $U \subset \tilde{U}$ of $P_{\tilde{q}}$ such that if $D > 0$ is the diameter of U , then $B^+(p, 2D) \cup B^-(p, 2D) \subset \tilde{U}$ for every $p \in U$. Let $q \in U$, P_q the plaque of $q \in U$, and $y_0, y_1 \in P_q$. Our goal is to check that $d(P_{\tilde{q}}, y_0) = d(P_{\tilde{q}}, y_1)$. This will imply $P_q \subset \mathcal{C}_r^+(P_{\tilde{q}})$.

For $i = 0, 1$, let $\gamma_i : [0, r_i] \rightarrow M$ be a minimal segment of geodesic joining $P_{\tilde{q}}$ with y_i , in particular $d(P_{\tilde{q}}, y_i) = \ell(\gamma_i)$, for $i = 0, 1$. Choose an arbitrary regular point $p \in M$. Then there exists a minimizing geodesic from L_p to \tilde{q} , whose singular points are isolated. As a consequence, we can consider a sequence of regular points p_m of the future tubular neighborhood U converging to \tilde{q} . Moreover, consider a sequence of minimal segments of geodesic β_i^m joining \tilde{P}_{p_m} to $\gamma_i(0)$ (even if \tilde{P}_{p_m} is not necessarily closed, the minimizing geodesic β_i^m exists

because $p_m \in \bar{B}^-(\gamma_i(0), D) \subset \tilde{U}$ by the choice of U). In particular,

$$d(\tilde{P}_{p_m}, \gamma_i(0)) = \ell(\beta_i^m).$$

Finally, consider minimal segments of geodesic $\gamma_i^m : [0, r_m] \rightarrow M$ joining \tilde{P}_{p_m} to y_i , which exist by the same reason as the β_i^m 's. It follows that

$$\ell(\gamma_i^m) = d(\tilde{P}_{p_m}, y_i) \quad \text{and} \quad \ell(\gamma_i^m) \leq \ell(\gamma_i \star \beta_i^m),$$

where \star stands for the concatenation of the two curves.

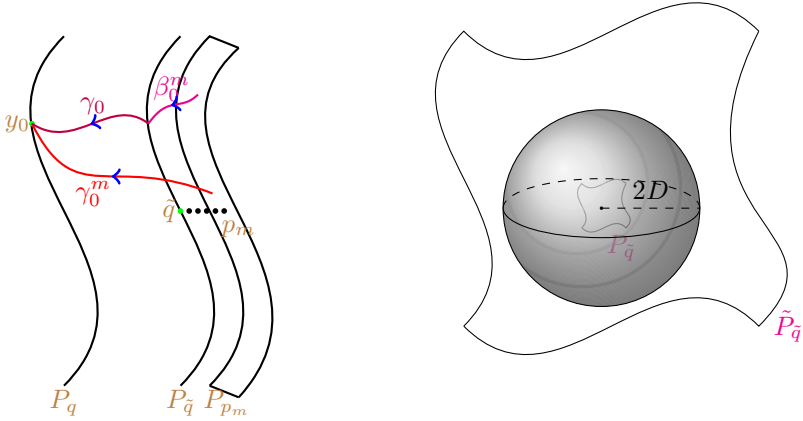


Figure 4. The diagram to the left represents the different curves used in the proof of Lemma 4.14. The diagram to the right depicts both plaques $P_{\tilde{q}}$ and $\tilde{P}_{\tilde{P}}$. The minimizing curves inside the tubular neighborhood U exist as short curves and cannot reach the boundary, because of the hypothesis on the diameter.

Observe that the image of the curves γ_i^m is contained in $\bar{B}^-(y_i, D)$. As a consequence, there exists a convergent subsequence $\gamma_i^{n_k}(0)$ converging to some $\hat{p} \in \tilde{P}_{\tilde{q}} \subset \tilde{U}$ (recall that the plaques \tilde{P}_{p_m} in U must necessarily converge to $\tilde{P}_{\tilde{q}}$). As all the sequence and the limit lie in the convex neighborhood \mathcal{U} , we can guarantee that $\gamma_i^{n_k}$ converges to a segment of geodesic $\hat{\gamma}_i$ joining $\tilde{P}_{\tilde{q}}$ to y_i . By Lemma 4.13, $\ell(\beta_i^m) \rightarrow 0$ when $m \rightarrow \infty$, and hence we conclude that $\ell(\hat{\gamma}_i) \leq \ell(\gamma_i)$. Since $d(\tilde{P}_{\tilde{q}}, y_i) = d(P_{\tilde{q}}, y_i) = \ell(\gamma_i)$, we infer that $\ell(\hat{\gamma}_i) = \ell(\gamma_i)$.

On the other hand, as the points p_m are regular, one can apply the same techniques of parallel transport of the proof of Lemma 4.13 to prove that $\ell(\gamma_0^m) = \ell(\gamma_1^m)$ and hence $\ell(\hat{\gamma}_1) = \ell(\hat{\gamma}_0)$. Therefore $\ell(\gamma_0) = \ell(\gamma_1)$ and this concludes the proof of the lemma. \blacksquare

LEMMA 4.15: *Let x be a point of a singular leaf $L_{\bar{q}}$. Then the geodesic γ_{ξ_x} is horizontal, i.e., orthogonal to the leaves of the singular foliation $\mathcal{F} = \{\pi^{-1}(c)\}$.*

Proof. The proof follows directly from Lemmas 4.14 and 2.5. \blacksquare

Remark 4.16: It is natural to wonder if it is possible to remove the assumption of being a smooth singular foliation in part (c) of Theorem 1.1. One possibility to face this problem is using the techniques of [33], since we have already proved the analog of [33, Proposition 26] in Lemma 4.6 and the analog of [33, Proposition 27] in Lemma 4.9. Furthermore, to obtain the analog of [33, Proposition 28] one should deal with completions of non-symmetric distances (see [16]). One of the most challenging parts could be to prove that $\eta_\xi^r : \pi^{-1}(c) \rightarrow \pi^{-1}(d)$ is surjective (the analog of [33, Proposition 29]) without assuming that \mathcal{F}_π is a singular foliation (recall Lemma 4.12). This fact could be important to show that (the analog of) the extension map obtained in [33, Theorem 25] can be identified with π .

5. On Jacobi fields and curvature in Finsler Geometry

5.1. ANISOTROPIC CONNECTION AND CURVATURE. In this section we review a few facts about anisotropic connections; see [24, 25].

As the tensors associated with Finsler metrics depend on the direction $v \in TM \setminus \mathbf{0}$, it is necessarily a connection that takes into account this fact, namely, it depends on the direction. In the following, we will denote by $\mathfrak{X}(M)$ the space of smooth vector fields on M and by $\mathcal{F}(M)$ the subset of smooth real functions on M .

Definition 5.1: An **anisotropic connection** is a map

$$\begin{aligned}\nabla : A \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\rightarrow TM, \\ (v, X, Y) &\mapsto \nabla_X^v Y \in T_p M \setminus \{0\},\end{aligned}$$

with $v \in T_p M$, where $A = TM \setminus \mathbf{0}$, such that for any $X, Y \in \mathfrak{X}(M)$, the map $A \ni v \rightarrow \nabla_X^v Y \in TM$ is smooth, and for all $p \in M$ and $v \in T_p M \setminus \{0\}$,

- (i) $\nabla_X^v(Y + Z) = \nabla_X^v Y + \nabla_X^v Z$, for any $X, Y, Z \in \mathfrak{X}(M)$,
- (ii) $\nabla_X^v(fY) = X_p(f)Y_p + f(p)\nabla_X^v Y$ for any $X, Y \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$,
- (iii) $\nabla_{fX+hY}^v Z = f(p)\nabla_X^v Z + h(p)\nabla_Y^v Z$, for any $X, Y, Z \in \mathfrak{X}(M)$ and $f, h \in \mathcal{F}(M)$.

Observe that the properties (ii) and (iii) imply, respectively, that $\nabla_X^v Y$ depends only on the value of Y on a neighborhood of p , and on the value of X_p . So sometimes we will put only X_p and it will make sense to compute $\nabla_X^v Y$ in a chart. If $(\Omega, \varphi = (x^1, \dots, x^n))$ is a chart of M , with its associated partial vector fields denoted by $\partial_1, \dots, \partial_n$, the **Christoffel symbols** of ∇ are the functions $\Gamma_{ij}^k : T\Omega \setminus \mathbf{0} \rightarrow \mathbb{R}$, $i, j, k = 1, \dots, n$, determined by

$$(5.1) \quad \nabla_{\partial_i}^v \partial_j = \sum_{k=1}^n \Gamma_{ij}^k(v) \partial_k,$$

for $i, j = 1, \dots, n$. Given a vector field V without singularities on an open set $\Omega \subset M$, the anisotropic connection ∇ induces an affine connection ∇^V on Ω defined as $(\nabla_X^V Y)_p = \nabla_{\tilde{X}}^{V_p} \tilde{Y}$ for any $X, Y \in \mathfrak{X}(\Omega)$, where $\tilde{X}, \tilde{Y} \in \mathfrak{X}(M)$ are extensions of X and Y , respectively, namely, they coincide with X and Y in a neighborhood of $p \in \Omega$. Moreover, if we define the **vertical derivative** of ∇ as the $(1, 3)$ -anisotropic tensor P given by

$$P_v(X, Y, Z) = \frac{\partial}{\partial t} (\nabla_X^{v+tZ_p} Y)|_{t=0},$$

for $p \in M$, $v \in T_p M \setminus \{0\}$ and $X, Y, Z \in \mathfrak{X}(M)$, then it is possible to define the curvature tensor of the anisotropic connection as

$$(5.2) \quad R_v(X, Y)Z = (R^V)_p(X, Y)Z - (P_V)_p(Y, Z, \nabla_X^V V) + (P_V)_p(X, Z, \nabla_Y^V V),$$

where $V \in \mathfrak{X}(\Omega)$ is an extension of $v \in T_p M \setminus \mathbf{0}$, $X, Y, Z \in \mathfrak{X}(M)$, P_V is the tensor such that $(P_V)_p = P_{V_p}$ and R^V is the curvature tensor of the affine connection ∇^V . It is important to observe that the curvature tensor R_v does not depend on the vector fields V, X, Y, Z chosen to make the computation, but only on v, X_p, Y_p and Z_p , [25, Prop. 2.5]. With an anisotropic connection at

hand, we can also compute the tensor derivative of any anisotropic tensor (see [24, §3]), and this computation can be done using ∇^V (see [24, Remark 15]). In most of the references of Finsler Geometry, they use a connection along the vertical bundle of TM . For a relationship between both types of connections see [24, §4.4].

5.2. A FEW PROOFS ABOUT FINSLER JACOBI FIELDS. Here, for the sake of completeness, we present the proof of a few results about Jacobi fields that we have used.

5.2.1. Proof of Lemma 3.4. Consider $J_1, J_2 \in \mathbb{W}$. Using the shape operator $\mathcal{S}_{\dot{\gamma}(t_0)}$ (see Definition 2.1), it follows that

$$\begin{aligned} g_{\dot{\gamma}(t_0)}(J'_1(t_0), J_2(t_0)) &= g_{\dot{\gamma}(t_0)}(\mathcal{S}_{\dot{\gamma}(t_0)}(J_1(t_0)), J_2(t_0)) \\ &= g_{\dot{\gamma}(t_0)}(J_1(t_0), \mathcal{S}_{\dot{\gamma}(t_0)}(J_2(t_0))) = g_{\dot{\gamma}(t_0)}(J_1(t_0), J'_2(t_0)), \end{aligned}$$

using that $\mathcal{S}_{\dot{\gamma}(t_0)}$ is self-adjoint (see [27, Prop. 3.5]). As we saw in Remark 3.2 this implies that \mathbb{W} is self-adjoint for all t .

In order to see that the dimension of \mathbb{W} is equal to $n - 1$, observe first that the dimension of the L -Jacobi fields is n . Recall that for every $t \in I$, if we define

$$\dot{\gamma}(t)^\perp := \{v \in T_{\dot{\gamma}(t)}M : g_{\dot{\gamma}(t)}(\dot{\gamma}(t), v) = 0\},$$

we have the splitting

$$(5.3) \quad T_{\dot{\gamma}(t)}M = \text{span}\{\dot{\gamma}(t)\} + \dot{\gamma}(t)^\perp,$$

and we will denote by $\tan_\gamma : \mathfrak{X}(\gamma) \rightarrow \mathfrak{X}(\gamma)$ and $\text{nor}_\gamma : \mathfrak{X}(\gamma) \rightarrow \mathfrak{X}(\gamma)$ the first and second projection of the splitting (5.3) at every $t \in I$. Observe that given an arbitrary Jacobi field J , then $J^T = \tan_\gamma J$ and $J^\perp = \text{nor}_\gamma J$ are again Jacobi fields along γ (see, for example, [27, Lemma 3.17]). Moreover, as J is L -Jacobi, then $J(t_0)$ is tangent to L and therefore $g_{\dot{\gamma}(t_0)}$ -orthogonal to $\dot{\gamma}(t_0)$. As a consequence, $J^T(t_0) = 0$. Taking into account that a Jacobi field is tangent to γ if and only if $J(t) = (a_1 t + a_2)\dot{\gamma}(t)$ (see [27, Lemma 3.17 (i)]), we conclude that $J^T(t) = a_1 t \dot{\gamma}(t)$, which is an L -Jacobi field and then J^\perp is also an L -Jacobi field, because $J^\perp = J - J^T$. Let $\mathcal{J}_L(\gamma)$ be the space of L -Jacobi fields along γ and $\mathcal{J}_L^T(\gamma) = \{J \in \mathcal{J}_L(\gamma) : J^\perp = 0\}$, $\mathcal{J}_L^\perp(\gamma) = \{J \in \mathcal{J}_L(\gamma) : J^T = 0\}$. Therefore, we have a map

$$\begin{aligned} \phi : \mathcal{J}_L^T(\gamma) \times \mathcal{J}_L^\perp(\gamma) &\rightarrow \mathcal{J}_L(\gamma), \\ (J_1, J_2) &\rightarrow J_1 + J_2, \end{aligned}$$

which is well-defined and one-to-one. As we have seen above that $\mathcal{J}_L^T(\gamma)$ has dimension one, it follows that $\mathcal{J}_L^\perp(\gamma)$ has dimension $n - 1$, which concludes the proof.

5.2.2. Proof of Proposition 3.5. Observe that the dimension of the submanifold $TL^\perp \subset TM$ of orthogonal vectors to L is $n = \dim M$ (see for example [27, Lemma 3.3]). Now consider a curve $(-\varepsilon, \varepsilon) \ni s \rightarrow N(s) \in TL^\perp$, with $\beta(s) = \rho(N(s))$ a curve in L such that $N(0) = \dot{\gamma}(t_0)$ (where $\rho : TM \rightarrow M$ is the natural projection). We can construct a variation $\Lambda(t, s) = \gamma_{N(s)}(t - t_0)$, $\Lambda : I \times (-\varepsilon, \varepsilon) \rightarrow M$, which is given by the L -orthogonal geodesics $\gamma_{N(s)}$ whose velocity at 0 is $N(s)$. Observe that

$$\Lambda(t_0, s) = \beta(s), \quad \frac{\partial \Lambda}{\partial t}(t_0, s) = N(s), \quad \frac{\partial \Lambda}{\partial s}(t_0, s) = \dot{\beta}(s).$$

It is not difficult to show that $V(t) = \frac{\partial \Lambda}{\partial s}(t, 0)$ is an L -Jacobi field along γ . Indeed, it is Jacobi, because its longitudinal curves are geodesics (see [27, Prop. 3.13]). In order to show that it is L -Jacobi, observe that $V(t_0) = \dot{\beta}(0)$ and if (Ω, φ) is a chart in a neighborhood of $\gamma(t_0)$, with N^i, β^i, γ^i , $i = 1, \dots, n$, the coordinates of N , β and γ , and $\Gamma_{jk}^i : T\Omega \setminus 0 \rightarrow \mathbb{R}$ the Christoffel symbols of the Chern connection in (Ω, φ) , then

$$(5.4) \quad V'(t_0) = D_{\dot{\gamma}}^{\dot{\gamma}} V(t_0) = D_{\dot{\beta}}^{\dot{\beta}} N(0) = (\dot{N}^i(0) + \dot{\beta}^j(0) \dot{\gamma}^k(t_0) \Gamma_{jk}^i(\dot{\gamma}(t_0))) \partial_i$$

(using [27, Eq. (7)] in the second equality). Therefore $V(t_0)$ is tangent to L and the second equality of (5.4) implies that $\tan_{\dot{\gamma}(t_0)}^L V'(t_0) = \mathcal{S}_{\dot{\gamma}(t_0)}(V(t_0))$, which concludes that V is L -Jacobi. In order to see that all the L -Jacobi fields can be obtained with variations of L -orthogonal geodesics, observe that the formula (5.4) allows us to define a map

$$\rho : T_{\dot{\gamma}(t_0)} TL^\perp \rightarrow T_{\gamma(t_0)} M \times T_{\gamma(t_0)} M,$$

defined as $\rho(\dot{N}(0)) = (V(t_0), V'(t_0))$. It is straightforward to check, using (5.4), that ρ is injective. Indeed, this follows from the expression of the coordinates of $\frac{dN}{ds}(0)$ in a natural chart of TTM associated with (Ω, φ) , which are $\gamma^1(0), \dots, \gamma^n(0), \dot{\gamma}^1(t_0), \dots, \dot{\gamma}^n(t_0), \dot{\beta}^1(0), \dots, \dot{\beta}^n(0), \dot{N}^1(0), \dots, \dot{N}^n(0)$, taking into account that, by definition, $N(0) = \dot{\gamma}(t_0)$ and $V(t_0) = \dot{\beta}(0)$. Moreover, as we have seen above, the image of ρ is contained in the initial values for the Cauchy problem of L -Jacobi fields. As the dimension of $T_{\dot{\gamma}(t_0)} TL^\perp$ coincides

with the dimension of the space of L -Jacobi fields (in both cases, the dimension is n), this easily implies that every L -Jacobi field can be realized as the variational vector field of a variation by L -orthogonal geodesics.

5.2.3. Proof of Lemma 3.6. The fact that (b) implies (a) follows from Proposition 2.3, because all the geodesics in the variation ψ project into the same geodesic on B , and as a consequence $J(t) = \frac{\partial}{\partial s}\psi(t, 0)$ is vertical for every $t \in I$. Let us check that (a) implies (b). First note that J is determined by its first initial condition $J(t_0)$, because J is vertical. In fact assume by contradiction that there exists another vertical L -Jacobi field \tilde{J} so that $\tilde{J}(t_0) = J(t_0)$ and $J \neq \tilde{J}$. Set $\hat{J} := J - \tilde{J}$. Hence \hat{J} is a vertical Jacobi field with $\hat{J}(t_0) = 0$, and then

$$\hat{J}'(t_0) = \lim_{t \rightarrow t_0} \frac{1}{t - t_0} \hat{J}(t),$$

which implies that $\hat{J}'(t_0)$ is also tangent to L . Being \hat{J} L -Jacobi (recall Definition 2.1), it follows that the tangent part to L of $\hat{J}'(t_0)$ is zero, which concludes that $\hat{J} = 0$, as both initial conditions are zero. Now consider a curve $\beta : (-\varepsilon, \varepsilon) \rightarrow L$ such that $\dot{\beta}(0) = J(t_0)$ and the Jacobi field $\tilde{J}(t) = \frac{\partial}{\partial s}\psi(t, 0)$, ψ being the variation defined in part (b). The fact that (a) implies (b) follows from the above discussion since $J(t_0) = \tilde{J}(t_0) = \frac{\partial}{\partial s}\psi(t, 0)$, and then $J = \tilde{J}$.

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